A Novel Approach in Geometrical-Mechanical Analysis of Plain Woven Fabrics; Initial Load-Extension Behavior

Mostafa Jamshidi Avanaki, Ali Asghar Asgharian Jeddi and Abbas Rastgoo

***Abstract*—A theoretical analysis for the initial state of load-extension behavior in plain woven fabrics is presented. For this purpose, a new approach for geometrical modeling of woven fabrics consisting of its structure in inclined and float regions is developed which results in theoretical estimation of all the structural parameters of plain fabrics including its weave angle. Then, by applying the strain energy method and considering a virtual spring in the unit cell, a modified model for predicting the initial tensile modulus of plain woven fabrics is proposed. The results are shown better agreement with experimental data than previous models.**

Key words: Load-extension, weave angle, strain energy, initial modulus

I. INTRODUCTION

The mechanical behavior of woven fabrics is of interest in numerous applications, including apparel and technical usages. Engineered designing of textile fabrics for specific mechanical properties requires the ability to predict its behavior in various loading conditions which among them, tensile properties has been analyzed by numerous researchers [1-7]. However, due to the geometrical model and the approach in mechanical analysis, most of the available works are resulted in complicated elliptic integrals or needs numerical solutions even in the domain of small deformation analysis.

There are numerous applications of woven fabrics involved with small extensions which have received much attention by some scientists [8-10]. The work developed by Leaf and Kandil [10] that considering this state of deformation is the basic approach in this work which is tractable and amenable to predict the initial load extension behavior of plain woven fabrics in a close form solution.

In this work, a new approach for the geometrical-mechanical analysis of the plain woven fabrics is developed which is also applicable to other woven structures.

It is assumed that the yarns cross section are always circular along the yarn path as was developed by Peirce [11] and the yarns center line obeys the saw tooth geometry which was developed by Kawabata [12]. On the base of these assumptions, a new geometrical modelling is proposed which includes both inclined and float regions. The mechanical approach in this work employs the strain energy method through applying the Castigliano's theorem. A virtual spring is defined in the unit cell of the structure and used in mechanical analysis which results in better prediction of initial tensile modulus of plain woven fabrics. The effect of friction is ignored in this work. It is shown that the proposed theory does lead to results that are in agreement with experimental data.

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II. GEOMETRICAL MODELLING

Geometrical modelling of woven structures has been the subject of many works [13-16]. Weaves geometry is determined by defining a unit cell representing the whole fabric characteristics. The proposed model and its structural parameters for the weave repeat of plain woven fabrics is shown in Figure 1. The indices i and j are used to denote warp and weft yarns sequentially and the indices f and c represent float and inclined regions respectively. The weave repeat in Figure 1 is shown for the weft yarns length which is included by the cross section of warp yarns. So, this structure can be also developed for warp lengthwise by substituting the index i instead of j. Moreover, Y_i is used to denote the spacing between warp yarns while P_j is the width of unit cell. In the unit cell, P_{fj} and P_{cj} are used to denote the projection of weft yarns float and inclined length, respectively. It should be noted here that in this work, all the lengths are considered to be in mm and the angels are in degree.

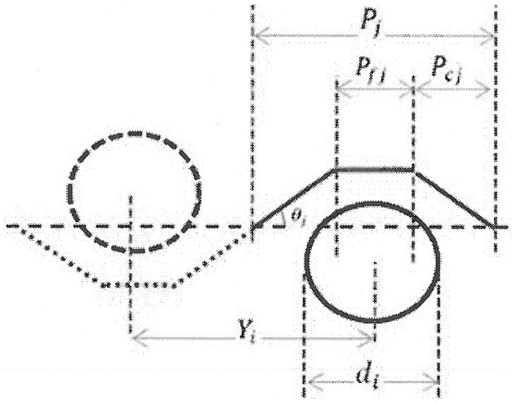


Fig. 1. Geometrical parameters in the weave repeat of plain fabrics.

The yarns diameter (d) is calculated by Eq. (1) given by Peirce [11].

$$d = \frac{\sqrt{Tex}}{28.02\sqrt{\Phi\rho_f}} \text{ (mm)} \quad (1)$$

In this equation, Tex is the yarn linear density (g/km), ρ_f is the fiber density (g/cm^3) and Φ is the yarn packing factor. For blended yarns, average fiber density is given by the Eq. (2).

$$\frac{1}{\rho_f} = \sum_{i=1}^{n} \frac{w_i}{\rho_i} \quad (2)$$

where; w_i is the weight fraction of the ith component, ρ_i is the fiber density of the ith component and n is the number of components of the blend. The proposed unit cell in our model is equivalent to the half of weave repeat which is demonstrated in Figure 1. Furthermore, due to symmetry, we consider half of the unit cell for developing the equations as is shown in Figure 2. Therefore, all the results will be doubled to cover the whole unit cell.

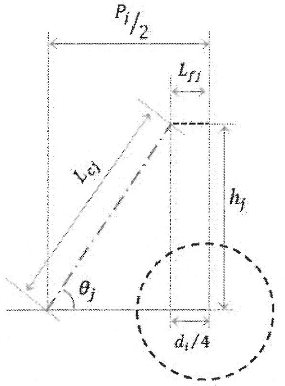


Fig. 2. Half of the Unit Cell Geometry.

In this geometry, L_{fj} is used to denote the weft length in float region and L_{cj} is used to denote its length in inclined region. Therefore, in the proposed geometrical model, the value of the total length of weft yarns in the unit cell can be obtained as follows:

$$L_j = 2(L_{fi} + L_{cj}) \quad (3)$$

In the proposed geometry, it is assumed that the projection length of weft yarns in float region (P_{fj}) is equal to the half of the warp yarns diameter which is obtained through Eq. (4). Therefore, by attention to Figure 1, the length of yarn in float region can be estimated by Eq. (5).

$$P_{fj} = \frac{d_i}{2} \quad (4)$$

$$L_{fj} = \frac{P_{fj}}{2} \quad (5)$$

Moreover, by knowing that the width of the proposed unit cell is equal to the warp yarns spacing, as shown by Eq. (6), the projection of weft yarns in inclined region (P_{cj}) can be obtained through Eq. (7).

$$P_j = Y_i \quad (6)$$

$$P_{cj} = \frac{P_j - P_{fj}}{2} \quad (7)$$

The amount of the length of weft yarns in the unit cell (L_{Oj}) and the spacing between warp yarns (Y_i) could be measured by experimental works. The amount of (L_{Oj}) represents the value of weft yarns length in unit cell which is differ from that in proposed straight line geometry due to the curvy path of yarns in real fabrics. So, its amount in the proposed geometry will be followed up here.

By knowing the length of yarn in float region (L_{fj}) through Eq. (5), the preliminary estimated length of weft yarn in crimp region (L_{Ocj}) can be obtained as follows:

$$L_{Ocj} = L_{Oj} - L_{fj} \quad (8)$$

It can be found through geometrical model that the projection of inclined yarn can be obtained as follows:

$$L_{cj} = L_{Ocj} \times \cos\theta_{Oj} = P_{cj} \quad (9)$$

So, the preliminary estimated value of weave angle can be calculated by Eq. (10).

$$\theta_{Oj} = \mathrm{Cos}^{-1}\left(\frac{P_{cj}}{L_{Ocj}}\right) \quad (10)$$

Moreover, the preliminary estimated value for half of the crimp amplitude (h_{0j}) can be calculated as follows:

$$h_{Oj} = L_{Ocj}\mathrm{Sin}\theta_{Oj} \quad (11)$$

After estimating the preliminary amounts of structural parameters, their modified values can be obtained. For this purpose, based on Peirce [11] assumption, the condition that the warp and weft yarns touch each other at cross over points is considered as constraint which have to be satisfied as follows in which h refers to the half of the crimp amplitude as is shown in Figure 2.

$$h_j + h_i = \frac{d_j}{2} + \frac{d_i}{2} \quad (12)$$

So, by considering Eqs. (13) and (14), the improved value of the weft yarns crimp amplitude (h_j) can be estimated as follows:

$$D_d = \frac{d_i + d_j}{2} \quad (13)$$

$$D_h = h_{Oi} + h_{Oj} \quad (14)$$

$$\nabla = \frac{D_d - D_h}{\text{K Ratio}} \quad (15)$$

$$h_j = h_{Oj} + \nabla \quad (16)$$

The amount of ∇ in Eq. (16) should be specified by applying an appropriate amount of K Ratio in Eq. (15). K Ratio represents the ratio of discrepancy between D_d and D_h which is allocated to crimp amplitude, (h_j). By the way, finally the modified value for weave angle is obtained as shown by Eq. (17).

$$\theta_j = \mathrm{Sin}^{-1}\left(\frac{h_j}{L_{Ocj}}\right) \quad (17)$$

By substituting this weave angle in Eq. (9), the improved value for the inclined yarn projection as (L''_{cj}) is calculated as follow:

$$\left(L''_{cj}\right) = L_{Ocj} \times Cos\left(\theta_j\right) \quad (18)$$

Difference between the projection lengths of inclined yarn in Eqs. (9) and (18) is related to the amount of yarns which is curved around each other at intersection points in real fabric, i.e.;

$$Q_j = \text{curved length} = L'_{cj} - L''_{cj} \quad (19)$$

This phenomenon is demonstrated in Figure 3.

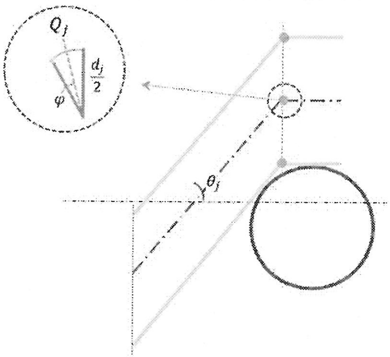


Fig. 3. Schematic demonstration of curved length.

The K Ratio index in Eq. (15) can be obtained by attention to the fact that some portion of difference between D_d and D_h in Eqs. (13) and (14) will be used to cover the curved shape in addition to satisfying the aforementioned constraints through Eq. (12). The optimum amount of K Ratio can be found through minimizing the following functions owing to the point that sum of the curved length (Q) and ∇ have to be equal or in very close agreement with the preliminary estimated discrepancy, $(D_d - D_h)$.

$$K_{i,j} = |Q| + |\nabla| - |D_d - D_h| \quad (20)$$

Equation (20) should be solved for both warp and weft yarns. In addition, it should be noted that the warp and weft yarns are assumed to be in equilibrium state of balanced fabrics. Therefore, the following function needs to be also minimized at the same time.

$$K = \left\| K_i \right| - \left| K_j \right\| \quad (21)$$

By applying Eqs. (20) and (21) simultaneously in a computer program (K programming), the optimum specific amount of K Ratio can be calculated. Then, the length of weft yarn in inclined region of the proposed unit cell is modified and obtained as follows:

$$L_{cj} = L_{Ocj} - Q_j \quad (22)$$

Therefore, by assuming straight line path for yarns in the unit cells, the length of weft yarns in the unit cell is obtained through Eq. (23).

$$L_j = 2\left(L_{fj} + L_{cj}\right) \quad (23)$$

III. MECHANICAL MODELLING

Mechanical modelling of woven structures has been the subject of many works including the work performed by Dabiryan [17] through defining frictional energy and other researches [18-20] through applying energy method and continuum mechanics.

The loading condition in the half of the unit cell is demonstrated in Figure 4. The force f_j is the external applied force. By applying the external tensile force, the compressive forces υ_j will be generated between yarns which are distributed over the region of yarn contact which was assumed to be pointed forces in previous works.

To predict the initial state of load-deformation behavior, it was assumed that in the field of loading condition, the yarns are linearly elastic; so the strain energy method and the Castigliano's theorem become applicable. Moreover, it was assumed that the fabric behave linearly elastic in its initial state of deformation; so it was possible to use the Hook's law.

By knowing that any possible deformation caused by applying forces which makes the yarns to be extended, bent, compressed or sheared, requires its own strain energy, the total required energy U_T for any deformation in the structure is obtained by using the following equation which is equal to sum of its components.

$$U_T + U_E + U_B + U_C + U_S \quad (24)$$

The components of strain energy in the unit cell constituents are proposed to be calculated as follows. For each component, the strain energy should be calculated in both inclined and float regions. In this work, the strain energy of yarns shearing which was ignored in previous works [10, 17] is considered.

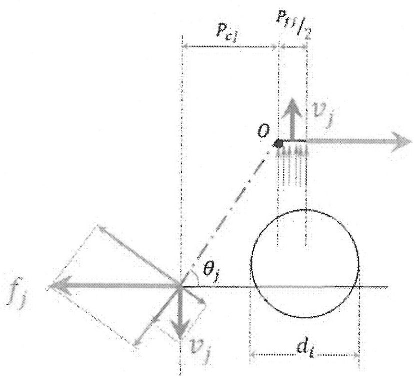


Fig. 4. Loading condition in the proposed geometry.

A. *Strain Energy Analysis*

The yarns are assumed to be linear elastic in the domain of loading conditions. So, the strain energy analysis can be applied on the elements of unit cell in inclined and float regions. Details are presented in Appendix I.

1) The Strain Energy in inclined Region

The strain energy of extension, bending, compression and shearing in inclined region is obtained as follows:

a) Strain Energy of Extension in inclined Region

$$T_{cj} = f_j Cos\theta_j + \upsilon_j Sin\theta_j$$

$$U_{ecj} = \left(\frac{1}{2\lambda_j} \right) . (f_i Cos\theta_j + \upsilon_j Sin\theta_j)^2 . L_{cj}$$

(1) Strain Energy of Bending in inclined Region

$$M_{cj} = (f_i \mathrm{Sin}\theta_j - \upsilon_j \mathrm{Cos}\theta_j) s$$

$$U_{bcj} = \left(\frac{1}{2B_j}\right) . (f_i \mathrm{Sin}\theta_j - \upsilon_j \mathrm{Cos}\theta_j)^2 . \left(\frac{L_{cj}^3}{3}\right)$$

(2) Strain Energy of Compression in inclined Region

$$V_{cj} = \eta_j \varepsilon_{dj} = 2(f_j \sin\theta_j - v_j \cos\theta_j)$$

$$U_{ccj} = \left(\frac{2}{\eta_j}\right) . (f_j \sin\theta_j - v_j \cos\theta_j)^2 . d_j . Q_j$$

(3) Strain Energy of Shearing in inclined Region

$$\tau_{cj} = (f_j \sin\theta_j - v_j \cos\theta_j)$$

$$U_{scj} = \left(\frac{1}{2G_j}\right) . (f_j \sin\theta_j - v_j \cos\theta_j)^2 . L_{cj}$$

b) The Strain Energy in Float Region

The strain energy of extension, bending, compression and shearing in float region is obtained as follows:

(1) Strain Energy of Extension in Float Region

$$T_{fj} = f_j$$

$$U_{efj} = \left(\frac{1}{2\lambda_j}\right) . f_j^2 . L_{fj}$$

(2) Strain Energy of Bending in Float Region

$$M_{fj} = v_j . s$$

$$U_{bfj} = \left(\frac{1}{2B_j}\right) . v_j^2 . \left(\frac{L_{fj}^3}{24}\right)$$

(3) Strain Energy of Compression in Float Region

$$V_{fj} = \eta_j \varepsilon_{dj} = 2v_j$$

$$U_{cfj} = \left(\frac{2}{\eta_j}\right) . v_j^2 . d_j . L_{fj}$$

(4) Strain Energy of Shearing in Float Region

$$\tau_{fj} = v$$

$$U_{sfj} = \left(\frac{1}{2G_j}\right) . v^2 . L_{fj}$$

B. Total Strain Energy

The total strain energy in the unit cell is calculated by using Equation (24) where;

$$U_E = U_{ec} + U_{ef} = 2(U_{eci} + U_{ecj} + U_{efi} + U_{efj})$$
$$U_B = U_{bc} + U_{bf} = 2(U_{bci} + U_{bcj} + U_{bfi} + U_{bfj})$$
$$U_C = U_{cc} + U_{cf} = 2(U_{cci} + U_{ccj} + U_{cfi} + U_{cfj})$$
$$U_s = U_{sc} + U_{sf} = 2(U_{sci} + U_{scj} + U_{sfi} + U_{sfj})$$

This leads to Equations (25) through (28) which will be used in calculation of fabric strain.

$$U_E = \frac{(f_j \cos\theta_j + v_j \sin\theta_j)^2 . L_{cj}}{\lambda_j} + \frac{(f_i \cos\theta_i + v_i \sin\theta_i)^2 . L_{ci}}{\lambda_i} + \frac{f_j^2 . L_{fj}}{\lambda_j} + \frac{f_i^2 . L_{fi}}{\lambda_i} \quad (25)$$

$$U_B = \frac{(f_j \sin\theta_j - v_j \cos\theta_j)^2 . L_{cj}^3}{3B_j} + \frac{(f_i \sin\theta_i - v_i \cos\theta_i)^2 . L_{ci}^3}{3B_i} + \frac{v_j^2 . L_{fj}^3}{24B_j} + \frac{v_i^2 . L_{fi}^3}{24B_i} \quad (26)$$

$$U_C = \frac{4(f_j \sin\theta_j - v_j \cos\theta_j)^2 . d_j . Q_j}{\eta_j} + \frac{4(f_i \sin\theta_i - v_i \cos\theta_i)^2 . d_i . Q_i}{\eta_i} + \frac{4v_j^2 . d_j . L_{fj}}{\eta_j} + \frac{4v_i^2 . d_i . L_{fi}}{\eta_i} \quad (27)$$

$$U_s = \frac{(f_j \sin\theta_j - v_j \cos\theta_j)^2 . L_{cj}}{G_j} + \frac{(f_i \sin\theta_i - v_i \cos\theta_i)^2 . L_{ci}}{G_i} + \frac{v_j^2 . L_{fj}}{G_j} + \frac{v_i^2 . L_{fi}}{G_i} \quad (28)$$

C. Calculation of fabric strain:

By knowing the total strain energy and applying the Castigliano's theorem, we know that

$$\varepsilon_j P_j = \delta P_j = \partial U_T / \partial f_j \quad (29)$$

Therefore the Equation (30) can be obtained.

$$\varepsilon_j P_j = \frac{2L_{cj}(f_j \cos\theta_j + v \sin\theta_j) . \cos\theta_j}{\lambda_j} + \frac{2L_{fj} f_j}{\lambda_j} + \frac{2L_{cj}^3 (f_j \sin\theta_j - v \cos\theta_j) . \sin\theta_j}{3B_j} + \frac{8(f_j \sin\theta_j - v_j \cos\theta_j) . d_j . Q_j . \sin\theta_j}{\eta_j} + \frac{2(f_j \sin\theta_j - v_j \cos\theta_j) . \sin\theta_j . L_{cj}}{G_j} \quad (30)$$

If the yarns are assumed to be inextensible and incompressible which can be correct just in the initial state of fabric deformation, so the λ , η and G tend to infinity and the Equation (30) reduces to:

$$\varepsilon_{j\,(\mathrm{Strian})} = \frac{f_j}{P_j}\left\{\frac{2L_{cj}^3 \sin^2\theta_j}{3B_j}\right\} - \frac{v}{P_j}\left\{\frac{2L_{cj}^3 \sin\theta_j . \cos\theta_j}{3B_j}\right\} \quad (31)$$

In this equation, the generated force between the yarns should be calculated.

$$\partial U_T / \partial v_j = \frac{\delta(h_j - D_j)}{2} \quad (32)$$

The amount of $\delta(h_j - D_j)$, denotes the change that occurs in the fabric thickness at cross-over points. When the fabric deformation takes place, the heights for the warp and weft yarns change but if the yarns are to remain in contact, due to crimp interchange, the changes must be equal. Therefore,

$\delta(h_i - D_i) = \delta(D_j - h_j)$

This gives;

$$\partial U_T/\partial v_i + \partial U_T/\partial v_j = 0$$

Therefore Eq. (33) can be obtained.

$$\sum_{k=i,j}\left\{\frac{2L_{ck}\,(f_k\cos\theta_k + v_k\sin\theta_k).\sin\theta_k}{\lambda_k} - \frac{2L_{ck}^3\,(f_k\sin\theta_k - v_k\cos\theta_k).\cos\theta_k}{3B_k} + \frac{v_k.L_{fk}^3}{12B_k} - \frac{8(f_k\sin\theta_k - v_k\cos\theta_k).d_k.Q_k.\cos\theta_k}{\eta_k} + \frac{8v_k.d_k.L_{fk}}{\eta_k} - \frac{2(f_k\sin\theta_k - v_k\cos\theta_k).\cos\theta_k.L_{ck}}{G_k} + \frac{2v_k.L_{fk}}{G_k}\right\} = 0 \quad (33)$$

Since, $v_i = v_j = v$ this equation can be solved for v to give

$$v = \frac{A}{B} \quad (34)$$

Where A and B are presented in Equations (35) and (36).

$$A = \sum_{k=i,j}\left\{f_k\cos\theta_k\sin\theta_k.\left(-\frac{2L_{ck}}{\lambda_k} + \frac{2L_{ck}^3}{3B_k} + \frac{8d_k.Q_k}{\eta_k} + \frac{2L_{ck}}{G_k}\right)\right\} \quad (35)$$

$$B = \sum_{k=i,j}\left\{\frac{2L_{ck}.\sin^2\theta_k}{\lambda_k} + \frac{2L_{ck}^3.\cos^2\theta_k}{3B_k} + \frac{L_{fk}^3}{12B_k} + \frac{8d_k.Q_k.\cos^2\theta_k}{\eta_k} + \frac{8d_k.L_{fk}}{\eta_k} + \frac{2L_{ck}.\cos^2\theta_k}{G_k} + \frac{2L_{fk}}{G_k}\right\} \quad (36)$$

Again the assumptions of inextensibility and incompressibility for the yarns are considered. So, Eq. (34) reduces to

$$v = \frac{f_i\cos\theta_i\sin\theta_i.\left(\frac{2L_{ci}^3}{3B_i}\right) + f_j\cos\theta_j\sin\theta_j.\left(\frac{2L_{cj}^3}{3B_j}\right)}{\frac{2L_{ci}^3.\cos^2\theta_i}{3B_i} + \frac{L_{fi}^3}{12B_i} + \frac{2L_{cj}^3.\cos^2\theta_j}{3B_j} + \frac{L_{fj}^3}{12B_j}} \quad (37)$$

D. *Calculation of initial fabric Young's modulus*

Eq. (31) represents the fabric strain in weft direction. If one assumes that, the behavior of fabric is linearly elastic in its initial state of deformation, it can be deduced that the Hook's law can be applied for calculation of fabric Young's modulus in such state. Moreover, in a uniaxial loading condition in weft direction, the amount of loads in warp direction (f_i) is equal to zero which results in theoretical estimation of initial fabric modulus as follows (detail are presented in Appendix II).

$$E_j = \frac{P_j}{P_i A_j\tan\theta_j}\left(1 + \frac{A_j.\cot\theta_j}{A_i.\cot\theta_i + \frac{L_{fi}^3}{12B_i} + \frac{L_{fj}^3}{12B_j}}\right) \quad (38)$$

where;

$$A_j = \frac{(2L_{cj}^3\sin\theta_j\cos\theta_j)}{3B_j} \quad (39)$$

$$A_i = \frac{(2L_{ci}^3\sin\theta_i\cos\theta_i)}{3B_i} \quad (40)$$

Equation (38) is the preliminary proposed theory for predicting the initial uniaxial modulus of plain woven structures in the linearly elastic state of deformation.

IV. EXPERIMENTAL

For the validation of the proposed theory, the experimental data reported by Leaf [10] was used. The specifications of samples are shown in Tables I and II for the warp and weft yarns, respectively.

The amounts of fiber density and yarn packing factor extracted from literature for the warp and weft yarns were used in Eq. (1) to calculate the yarn diameter. The results are shown in Tables III and IV.

TABLE I
WARP YARN SPECIFICATION [10]

Fabric group	Yarn linear density (tex)	Material	Spinning method	Twist (turns/cm)
Common to all	60/2	polyester	not-known	not-known

TABLE II
WEFT YARN SPECIFICATIONS [10]

Fabric group	Yarn linear density (tex)	Material	Spinning method	Twist (turns/cm)
X	60/2	cotton	ring	6
Y	74/2	cotton	ring	5.2
Z	98/2	cotton	open end	4.4
A	60/2	polyester	not-Known	4.0
B	60/2	polyester-cotton	not-Known	4.2
C	46/2	polyester-cotton	not-Known	7.1

TABLE III
WARP YARN CALCULATED PARAMETERS

Fabric group	Fiber density $\rho f\left(gr/cm\right)$	Yarn packing factor ϕ	d_i (mm)
Common to all	1.38	0.65	0.292

TABLE IV
WEFT YARN CALCULATED PARAMETERS

Fabric group	Fiber density $\rho f\left(gr/cm\right)$	Yarn packing factor ϕ	d_i (mm)
X	1.52	0.6	0.289
Y	1.52	0.58	0.327
Z	1.52	0.55	0.386
A	1.38	0.65	0.292
B	1.45	0.62	0.292
C	1.45	0.64	0.251

The warp and weft yarn spacing, the length of yarns in the unit cell and the yarns bending rigidity are shown in Table V. By applying the proposed theory for geometrical modeling of the structure, the geometrical parameters of samples were theoretically estimated in both warp and weft directions. Results for the warp direction are represented in Table VI. The amount of K Ratio was calculated by applying Eq. (20).

Finally, the preliminary amounts of initial tensile modulus of samples were calculated through applying the Eq. (38) for the warp yarns. The results are shown in Table VII with the results reported by Leaf [10] for both measured values through experimental works and calculated values through applying his theory.

By comparing the results, it is found that the outputs of preliminary proposed model is far from experimental Leaf's values which indicate that some modifications have to be performed. This deviation seems to be strongly due to the fact that the inclined and float regions in the proposed geometry are not simply jointed without any effect on each other. Therefore, it was assumed that these elements are jointed together through a virtual spring (elastic rod spring as is demonstrated in Figure 3) and the strain energy of this spring was calculated and considered in the mechanical modelling as follows.

TABLE V
EXPERIMENTAL SPECIFICATIONS OF SAMPLES [10]

Fabric group	Sample No.	Yarn spacing (mm)		Unit cell yarn length (mm)		Bending rigidity ($mN.mm^2$)	
		Y_i	Y_j	L_{oi}	L_{oj}	B_i	B_j
	1	0.485	0.588	0.700	0.514	5.62	6.06
X	2	0.488	0.624	0.758	0.515	5.62	6.06
	3	0.485	0.713	0.835	0.508	5.62	6.06
	1	0.490	0.677	0.798	0.514	5.62	7.05
Y	2	0.492	0.739	0.871	0.515	5.62	7.05
	3	0.495	0.849	0.983	0.513	5.62	7.05
	1	0.494	0.779	0.939	0.508	5.62	8.16
Z	2	0.494	0.839	1.022	0.507	5.62	8.16
	3	0.491	0.691	0.847	0.509	5.62	8.16
	1	0.476	0.589	0.704	0.504	4.44	4.44
A	2	0.587	0.794	0.827	0.616	4.44	4.44
	3	0.549	0.532	0.606	0.615	4.44	4.44
	1	0.556	0.548	0.598	0.623	4.44	4.25
B	2	0.591	0.637	0.722	0.622	4.44	4.25
	3	0.594	0.756	0.832	0.624	4.44	4.25
	1	0.568	0.465	0.509	0.621	4.44	2.96
C	2	0.577	0.538	0.597	0.639	4.44	2.96
	3	0.571	0.662	0.730	0.608	4.44	2.96

V. MODIFIED PROPOSED THEORY

To consider the strain energy of virtual spring in the unit cell, it was first necessary to define its rigidity. For this purpose Γ was considered to be the generated torque in the spring through applied force (f) at distance (r) as follows:

$$\Gamma = f.r$$

Then by using Castigliano's theorem through strain energy method, and considering the virtual spring diameter as D_s and the number of its helix rings as N_s, the resulting angle after applying the external torque Γ is obtained as follows. The strain energy in virtual spring is calculated by:

$$U_\Lambda = \int_0^{\pi D_s N_s} \frac{\Gamma^2}{2B} ds$$

Applying the Castigliano's theorem, gives;

$$r.\theta = \frac{\partial U_\Lambda}{\partial f} = \int_0^{\pi D_s N_s} \frac{\partial}{\partial f}\left(\frac{f^2.r^2\, ds}{2B}\right) = \int_0^{\pi D_s N_s} \frac{f.r^2\, ds}{B}$$

Leading to;

$$\theta = \frac{\pi D_s N_s \Gamma}{B} \tag{41}$$

TABLE VI
CALCULATED GEOMETRICAL PARAMETERS IN WARP DIRECTION

Fabric group	Sample no.	K Ratio	θ_i (degree)	P_{fi} (mm)	P_{ci} (mm)	L_{fi} (mm)	L_{ci} (mm)	Q_i (mm)
	1	1.71	45.36	0.1447	0.2216	0.0724	0.2511	0.0266
X	2	1.67	43.05	0.1447	0.2396	0.0724	0.2911	0.0156
	3	1.59	38.66	0.1447	0.2841	0.0724	0.3305	0.0146
	1	1.67	44.13	0.1635	0.2568	0.0817	0.2882	0.0290
Y	2	1.62	40.59	0.1635	0.2878	0.0817	0.3346	0.0191
	3	1.54	36.34	0.1635	0.3428	0.0817	0.3971	0.0127
	1	1.65	45.56	0.1932	0.2929	0.0966	0.3411	0.0318
Z	2	1.6	42.39	0.1932	0.3229	0.0966	0.3976	0.0168
	3	1.74	50.60	0.1932	0.2489	0.0966	0.2855	0.0414
	1	1.72	45.90	0.1459	0.2215	0.0730	0.2517	0.0273
A	2	1.53	34.42	0.1459	0.3015	0.0730	0.3199	0.0206
	3	1.73	41.42	0.1459	0.1930	0.0730	0.2095	0.0205
	1	1.7	39.05	0.1458	0.2011	0.0729	0.2006	0.0255
B	2	1.6	40.21	0.1458	0.2456	0.0729	0.2625	0.0256
	3	1.53	33.74	0.1458	0.3051	0.0729	0.3233	0.0198
	1	1.69	43.76	0.1256	0.1697	0.0628	0.1604	0.0313
C	2	1.65	35.55	0.1256	0.2062	0.0628	0.2213	0.0144
	3	1.55	33.36	0.1256	0.2682	0.0628	0.2864	0.0158

Therefore the deformation rigidity of the virtual spring in the structure can be obtained through Eq. (42). In this equation, instead of helix rings number (N_s), two cases were assumed. In the first case, the contact length between warp and weft yarns extracted from geometrical model was used while in the second case the curved length was considered.

$$\Lambda_j = \frac{\Gamma_j}{\theta_j} = \frac{B_j}{\pi D_s N_s} = \frac{B_j}{\pi D_s \frac{\text{spring length}}{\pi D_s}} \tag{42}$$

$$\Lambda_{1j} = \frac{B_j}{Q_j + L_{fj}}$$

$$\Lambda_{2j} = \frac{B_j}{Q_j}$$

By attention to the proposed geometrical model, the torque in the structure is obtained by using Eq. (43) as follows:

$$\Gamma_j = (f_j \sin\theta_j - v_j \cos\theta_j).L_{cj}.W_j \tag{43}$$

Where by attention to the above mentioned cases, W_j refers to W_{1j} and W_{2j} in which;

$$W_{1j} = \left(1 + \frac{Q_j + L_{fj}}{L_{cj}}\right)$$

$$W_{2j} = \left(1 + \frac{Q_j}{L_{cj}}\right)$$

TABLE VII
INITIAL TENSILE MODULUS OF SAMPLES

Fabric group	Sample no.	E_i (mN/cm) Experimental	Leaf's theory	Preliminary theory	Modified theory Case 1	Modified theory Case 2
	1	14.3	25.57	28.87	9.55	22.94
X	2	9.4	22.08	26.83	10.29	24.00
	3	14.2	30.37	36.90	14.38	33.44
	1	15.9	30.86	28.57	10.51	23.46
Y	2	15.5	31.67	31.67	12.98	28.32
	3	14.6	38.69	41.14	17.81	38.54
	1	13.7	31.62	22.04	9.47	19.09
Z	2	10.6	31.07	24.33	11.63	22.92
	3	14.9	27.02	18.68	6.83	14.51
	1	9.2	19.77	25.95	7.67	20.15
A	2	9.1	21.30	26.41	9.52	21.89
	3	12.7	19.48	26.39	7.70	20.22
	1	24.0	29.00	34.54	8.96	23.86
B	2	13.3	15.77	19.27	6.32	15.05
	3	11.7	20.71	25.83	9.43	21.52
	1	23.2	27.32	34.02	6.77	19.42
C	2	18.0	19.16	25.81	8.84	21.31
	3	12.0	18.56	24.92	9.17	20.97

Therefore the strain energy required for its deformation can be calculated by Eq. (44).

$$U_{\Lambda j} = \left({}^{1}/_{2\Lambda_j}\right).\left((f_j \sin\theta_j - v_j \cos\theta_j).L_{cj}.W_j\right)^2 \quad (44)$$

Which its amount for the warp and weft yarns is

$$U_{\Lambda} = 2U_{\Lambda j} + 2U_{\Lambda i} \quad (45)$$

Now, it is possible to include the strain energy of virtual spring into the total strain energy formulation of the unit cell as follows:

$$U_T = U_E + U_B + U_C + U_s + U_{\Lambda} \quad (46)$$

Finally, by following the same procedure presented earlier, the modified theory is proposed as follows (details are presented in Appendix III).

$$E_j = \frac{P_j}{P_i(A_j)\tan\theta_j}\left\{1 + \frac{A_j \cot\theta_j}{A_i \cot\theta_i + \left(\frac{L_{fi}^3}{12B_i} + \frac{L_{fj}^3}{12B_j}\right)}\right\} \quad (47)$$

Where;

$$A_j = 2\,L_{cj}^2 \sin\theta_j \cos\theta_j.\left(\frac{L_{cj}}{3B_j} + \frac{W_j^2}{\Lambda_j}\right) \quad (48)$$

$$A_i = 2\,L_{ci}^2 \sin\theta_i \cos\theta_i.\left(\frac{L_{ci}}{3B_i} + \frac{W_i^2}{\Lambda_i}\right) \quad (49)$$

This theory was again applied to the samples and the results are shown in Table VII. The results are in close agreement with Leaf's proposed theory. The discrepancy between results extracted from theoretical models and experiments are shown in Table VIII. The deviation between predicted results and experimental values seems to be due to the fact that the cross sections in samples are not circular which was assumed in proposed model. Moreover, the predicted values in modified theories are found to be in better agreement with experimental values than preliminary theory in second case while the first case of modified theory correspond well with experimental values just for samples in groups X, Y and Z.

TABLE VIII
THE COMPARISON OF RESULTS FOR THE TENSILE MODULUS

Fabric group	Sample no.	Experimental E_i (mN/cm)	Discrepancy (%) Leaf's theory	Preliminary theory	Modified theory Case 1	Modified theory Case 2
	1	14.3	44.08	50.47	49.80	37.66
X	2	9.4	57.44	64.96	8.64	60.83
	3	14.2	53.24	61.51	1.27	57.53
	1	15.9	48.48	44.34	51.26	32.24
Y	2	15.5	51.05	51.06	19.39	45.26
	3	14.6	62.26	64.51	18.03	62.12
	1	13.7	56.67	37.83	44.63	28.24
Z	2	10.6	65.89	56.43	8.87	53.75
	3	14.9	44.85	20.26	118.03	2.67
	1	9.2	53.46	64.55	19.97	54.35
A	2	9.1	57.29	65.54	4.43	58.42
	3	12.7	34.80	51.88	64.88	37.21
	1	24.0	17.25	30.51	167.96	0.61
B	2	13.3	15.68	30.98	110.36	11.63
	3	11.7	43.52	54.70	24.08	45.62
	1	23.2	15.08	31.80	242.59	19.44
C	2	18.0	6.03	30.26	103.60	15.53
	3	12.0	35.34	51.85	30.81	42.78

Results of experimental and proposed theoretical model are shown in Figure 5 indicating that considering the virtual spring as the joining elements in the structure, leads to improvement of the final predicted values.

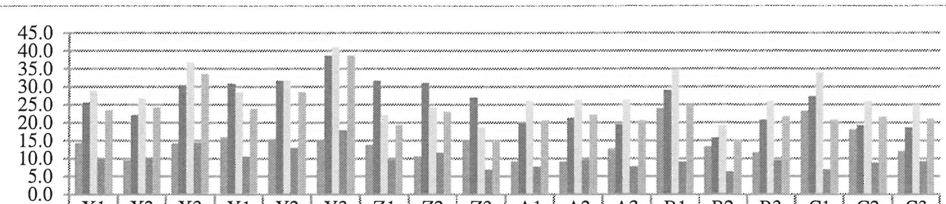
VI. CONCLUSION

In this work, a new approach for geometrical-mechanical modeling of plain woven fabrics was developed. In the proposed geometrical model, it was assumed that the yarns have two regions (inclined and float) in their path. By applying this model and an appropriate amount of K Ratio, all the structural parameters including the weave angle and curved length were theoretically calculated.

Then, the strain energy method and Castigliano's theorem were used for the mechanical analysis of the structure. The resulted theory was used in predicting of initial tensile modulus of some samples which were in large deviation from experimental data. Then, the theory was modified by assuming a virtual spring in the unit cell. For this purpose, the calculated amount of contact and curved lengths was used as first and second cases, respectively. The resulting modified theories were again checked with same samples.

This shows that in the case of saw tooth geometry, the inclined and float elements in the unit cell structure are not simply jointed in a point meaning that these regions have mutual effect on each other which is in accordance with reality.

Therefore a theoretical model for predicting the initial modulus of plain woven fabrics was developed which considered the inclined and float regions for its unit cell



	X1	X2	X3	Y1	Y2	Y3	Z1	Z2	Z3	A1	A2	A3	B1	B2	B3	C1	C2	C3
Experiment	14.3	9.4	14.2	15.9	15.5	14.6	13.7	10.6	14.9	9.2	9.1	12.7	24.0	13.3	11.7	23.2	18.0	12.0
Leaf Theory	25.6	22.1	30.4	30.9	31.7	38.7	31.6	31.1	27.0	19.8	21.3	19.5	29.0	15.8	20.7	27.3	19.2	18.6
Preliminary Theory	28.9	26.8	36.9	28.6	31.7	41.1	22.0	24.3	18.7	26.0	26.4	26.4	34.5	19.3	25.8	34.0	25.8	24.9
Modified Theory 1st Case	9.5	10.3	14.4	10.5	13.0	17.8	9.5	11.6	6.8	7.7	9.5	7.7	9.0	6.3	9.4	6.8	8.8	9.2
Modified Theory 2nd Case	23.4	24.1	33.6	23.8	28.5	38.6	19.3	23.0	14.9	20.6	22.1	20.6	24.7	15.3	21.7	20.7	21.5	21.2

Fig. 5. The results for the initial tensile modulus of plain woven fabrics (mN/cm)

structure. This was an extra advantage of the proposed model which made it capable to be extended for other woven structures which are our aims in future works.

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NOTATIONS

Subscript i	Denotes to the values in warp direction
Subscript j	Denotes to the values in weft direction
ρ_i	fiber density of the i[th] component of the bleneded yarn
ρ_{fiber}	fiber density (g/cm^3)
$\emptyset$	yarn packing factor
Λ_j	Deformation rigidity of the virtual spring
λ_j	Elastic tensile rigidity of weft yarns
η_j	Elastic compression rigidity of weft yarns
ε_j	Strain in fabric along the weft direction
Γ_j	the generated torque in the unit cell
τ_{cj}	Shearing force on weft yarns cross-section in inclined region
τ_{fj}	Shearing force on weft yarns cross-section in float region
B_j	Elastic bending rigidity of weft yarns
d_i	Diameter of warp yarns (mm)
D_d	Sum of warp and weft yarns diameter (mm)
D_h	Sum of warp and weft yarns amplitude (mm)
D_s	diameter of the virtual spring (mm)
E_j	Initial tensile modulus of fabric along the weft direction
F_j	Tensile force per unit width along the weft direction
f_j	Tensile force per individual weft yarns
G_j	Elastic shearing rigidity of weft yarns
h_{0j}	Preliminary estimated value for the weft yarns half inclined amplitude (mm)
h_j	weft yarns half crimp amplitude (mm)
K Ratio	The ratio of discrepancy between D_1 and D_2

	which is allocated to crimp amplitude
L_{0j}	Measured value of the total length of yarns in the unit cell (mm)
L_j	Total length of weft yarns in the unit cell (mm)
L_{fj}	Weft length in float region of the unit cell (mm)
L_{0cj}	Preliminary estimated length of weft yarn in inclined region (mm)
L_{cj}	Skewed weft length in inclined region of the unit cell (mm)
L'_{cj}	Length of inclined yarn projection (mm)
L''_{cj}	Modified value for the length of inclined yarn projection (mm)
M_{cj}	Bending force perpendicular to weft yarn axis in inclined region
M_{fj}	Bending force perpendicular to weft yarn axis in float region
N_s	number of helix rings in the virtual spring
P_j	Width of unit cell(mm)
P_{fj}	Projection of weft yarns float length (mm)
P_{cj}	Projection of weft yarns inclined length (mm)
Q	Curved length of yarns (mm)
Tex	Thread linear density (gram/km)
T_{cj}	Tensile force along weft yarn axis in inclined region
T_{fj}	Tensile force along weft yarn axis in float region
U_T	Total Strain Energy in the unit cell
U_E	Strain Energy of Extension in the unit cell
U_B	Strain Energy of Bending in the unit cell
U_C	Strain Energy of Compression in the unit cell
U_s	Strain Energy of Shearing in the unit cell
U_Λ	Strain Energy of torque in the unit cell
U_{ecj}	Strain Energy of Extension in inclined region of weft yarns in the half unit cell
U_{bcj}	Strain Energy of bending in inclined region of weft yarns in the half unit cell
U_{ccj}	Strain Energy of Compression in inclined region of weft yarns in the half unit cell
U_{scj}	Strain Energy of Shearing in inclined region of weft yarns in the half unit cell
U_{efj}	Strain Energy of Extension in float region of weft yarns in the half unit cell
U_{bfj}	Strain Energy of bending in float region of weft yarns in the half unit cell
U_{cfj}	Strain Energy of Compression in float region of weft yarns in the half unit cell
U_{sfj}	Strain Energy of Shearing in float region of weft yarns in the half unit cell
$U_{\Lambda j}$	Strain Energy of torque in the weft yarns of the unit cell
v_i	The generated force between warp yarns of the unit cell
V_{cj}	Compressive force on weft yarns curved length in inclined region
V_{fj}	Compressive force on weft yarns curved length in float region
w_i	weight fraction of the i^{th} component in the bleneded yarn
Y_i	Spacing between centers of warp yarn (mm)

APPENDIX I

The Strain Energy in inclined Region:

Strain Energy of Extension

$$U_{ecj} = \left(1/2\lambda_j\right) \cdot \int_0^{L_{cj}} T_{cj}^2 \, ds$$

Strain Energy of Bending

$$U_{bcj} = \left(1/2B_j\right) \cdot \int_0^{L_{cj}} M_{cj}^2 \, ds$$

Strain Energy of Compression

$$U_{ccj} = \left(1/2\eta_j\right) \cdot \int_0^{Q_j} \left(\int_0^{d_j} V_{cj}^2 \, dy \right) ds$$

Strain Energy of Shearing

$$U_{scj} = \left(1/2G_j\right) \cdot \int_0^{L_{cj}} {\tau_j}^2 \, ds$$

The Strain Energy in Float Region:

Strain Energy of Extension

$$U_{efj} = \left(1/2\lambda_j\right) \cdot \int_0^{L_{fj}} T_{fj}^2 \, ds$$

Strain Energy of Bending

$$U_{bfj} = \left(1/2B_j\right) \cdot \int_0^{\frac{L_{fj}}{2}} M_{fj}^2 \, ds$$

Strain Energy of Compression

$$U_{cfj} = \left(1/2\eta_j\right) \cdot \int_0^{L_{fj}} \left(\int_0^{d_j} V_{fj}^2 \, dy \right) ds$$

Strain Energy of Shearing

$$U_{sfj} = \left(1/2G_j\right) \cdot \int_0^{L_{fj}} {\tau_j}^2 \, ds$$

APPENDIX II

$$f_j = F_j P_i \xrightarrow{\text{yields}} \frac{f_j}{F_j} = P_i$$

$$F_j = E_j \varepsilon_j \xrightarrow{\text{yields}} \varepsilon_j = \frac{F_j}{E_j}$$

$$\varepsilon_{j\,(\text{Strian})} = \frac{f_j}{P_j}\left\{\frac{2L_{cj}^3 \sin^2\theta_j}{3B_j}\right\} - \frac{v}{P_j}\left\{\frac{2L_{cj}^3 \sin\theta_j \,.\cos\theta_j}{3B_j}\right\}$$

$$\frac{1}{E_j} = \frac{P_i}{P_j}\left\{\frac{2L_{cj}^3 \sin^2\theta_j}{3B_j}\right\} - \frac{P_i}{P_j}\frac{v}{f_j}\left\{\frac{2L_{cj}^3 \sin\theta_j \,.\cos\theta_j}{3B_j}\right\}$$

In an uniaxial loading condition, $f_i = 0$. Therefore

$$\frac{1}{E_j} = \frac{P_i}{P_j}(A_j.\tan\theta_j) - \frac{P_i}{P_j}\frac{(A_j)^2}{\left(\frac{2L_{ci}^3 .\cos^2\theta_i}{3B_i} + \frac{L_{fi}^3}{12B_i} + \frac{2L_{cj}^3 .\cos^2\theta_j}{3B_j} + \frac{L_{fj}^3}{12B_j}\right)}$$

By some mathematical works, the proposed model is obtained as follows:

$$E_j = \frac{P_j}{P_i A_j \tan\theta_j}\left(1 + \frac{A_j.\cot\theta_j}{A_i.\cot\theta_i + \frac{L_{fi}^3}{12B_i} + \frac{L_{fj}^3}{12B_j}}\right)$$

Where;

$$A_j = \frac{(2L_{cj}^3 \sin\theta_j \cos\theta_j)}{3B_j}$$

$$A_i = \frac{(2L_{ci}^3 \sin\theta_i \cos\theta_i)}{3B_i}$$

Therefore by substituting index i instead of j in the above equations, the initial tensile modulus is obtained as follows:

$$E_i = \frac{P_i}{P_j A_i \tan\theta_i}\left(1 + \frac{A_i.\cot\theta_i}{A_j.\cot\theta_j + \frac{L_{fi}^3}{12B_i} + \frac{L_{fj}^3}{12B_j}}\right)$$

APPENDIX III

For the second case of modified theory:

$$\Gamma_j = (f_j \sin\theta_j - v_j \cos\theta_j).(L_{cj} + Q_j)$$

$$\Gamma_j = (f_j \sin\theta_j - v_j \cos\theta_j).L_{cj}.W_j$$

$$W_j = \left(1 + \frac{Q_j}{L_{cj}}\right)$$

Therefore the strain energy required for its deformation can be calculated as follows:

$$U_{\Lambda j} = \left({}^{1}/_{2\Lambda_j}\right).\Gamma_j^2$$

$$U_{\Lambda j} = \left({}^{1}/_{2\Lambda_j}\right)\left((f_j \sin\theta_j - v_j \cos\theta_j).L_{cj}.W_j\right)^2$$

The strain energy equation of virtual spring is as follows:

$$U_\Lambda = \left({}^{1}/_{\Lambda_j}\right)\left((f_j \sin\theta_j - v_j \cos\theta_j).L_{cj}.W_j\right)^2 + \left({}^{1}/_{\Lambda_i}\right)\left((f_i \sin\theta_i - v_i \cos\theta_i).L_{ci}.W_i\right)^2$$

If the yarns are assumed to be inextensible and incompressible, so the λ , η and G tend to infinity, this leads to the following equation for the strain in the structure:

$$\varepsilon_{j\,(\mathrm{Strian})} = \frac{f_j}{P_j}\left\{2L_{cj}^2 \sin^2\theta_j\left(\frac{L_{cj}}{3B_j} + \frac{W_j^2}{\Lambda_j}\right)\right\} - \frac{v}{P_j}\left\{2L_{cj}^2.\sin\theta_j \cos\theta_j\left(\frac{L_{cj}}{3B_j} + \frac{W_j^2}{\Lambda_j}\right)\right\}$$

By applying the procedure introduced before, the equation representing the forces between yarns considering the virtual spring is as Eq. (50).

$$\sum_{k=i,j}\left\{\frac{2L_{ck}(f_k \cos\theta_k + v_k \sin\theta_k).\sin\theta_k}{\lambda_k} - \frac{2L_{ck}^3(f_k \sin\theta_k - v_k \cos\theta_k).\cos\theta_k}{3B_k} + \frac{v_k.L_{fk}^3}{12B_k} - \frac{8(f_k \sin\theta_k - v_k \cos\theta_k).d_k.Q_k.\cos\theta_k}{\eta_k} + \frac{8v_k.d_k.L_{fk}}{\eta_k} - \frac{2(f_k \sin\theta_k - v_k \cos\theta_k).\cos\theta_k.L_{ck}}{G_k} + \frac{2v_k.L_{fk}}{G_k} - \frac{2L_{ck}^2 W_k^2.\cos\theta_k\,(f_k \sin\theta_k - v_k \cos\theta_k)}{\Lambda_k}\right\} = 0 \quad (50)$$

Since $v_i = v_j = v$, this equation can be solved for v which leads to Eq. (51) in an uniaxial test.

$$v = \frac{f_i \cos\theta_i \sin\theta_i.\left(\frac{2L_{ci}^3}{3B_i}\right) + f_j \cos\theta_j \sin\theta_j.\left(\frac{2L_{cj}^3}{3B_j}\right) + f_i \cos\theta_i \sin\theta_i.\left(\frac{2L_{ci}^2 W_i^2}{\Lambda_i}\right) + f_j \cos\theta_j \sin\theta_j.\left(\frac{2L_{cj}^2 W_j^2}{\Lambda_j}\right)}{\frac{2L_{ci}^3.\cos^2\theta_i}{3B_i} + \frac{L_{fi}^3}{12B_i} + \frac{2L_{cj}^3.\cos^2\theta_j}{3B_i} + \frac{L_{fj}^3}{12B_i} + \frac{2L_{ci}^2.W_i^2.\cos^2\theta_i}{\Lambda_i} + \frac{2L_{cj}^2.W_j^2.\cos^2\theta_j}{\Lambda_i}} \quad (51)$$

Considering;

$$A_j = 2\,L_{cj}^2 \sin\theta_j \cos\theta_j.\left(\frac{L_{cj}}{3B_j} + \frac{W_j^2}{\Lambda_j}\right)$$

Gives:

$$E_j = \frac{P_j}{P_i(A_j)\tan\theta_j}\left\{1 + \frac{A_j \cot\theta_j}{A_i \cot\theta_i + \left(\frac{L_{fi}^3}{12B_i} + \frac{L_{fj}^3}{12B_j}\right)}\right\}$$

Therefore by substituting index i instead of j in the above equations, the initial tensile modulus in warp direction can be obtained.